Chapter 7

Finite Volume Method

Second Session Contents:

Exponential Method

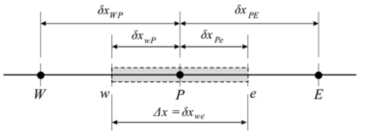
Consider 1-D convection-diffusion equation

$$\frac{d}{dx}\left(\rho u\phi - \Gamma\frac{d\phi}{dx}\right) = 0 \longrightarrow$$ This equation can be solved analytically

$$J = \rho u\phi - \Gamma\frac{d\phi}{dx} \longrightarrow \frac{dJ}{dx} = 0$$

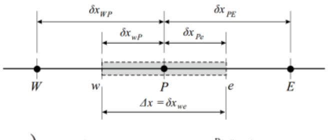
Integrating over cell element, we have

$$J_e - J_w = 0$$



Exponential Method

$$J_e = (\rho u)_e\,\phi_e - \Gamma_e\left(\frac{d\phi}{dx}\right)_e$$



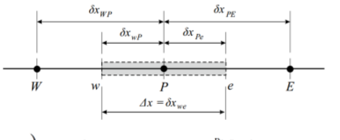
$$J_e = F_e\left(\phi_P + (\phi_E - \phi_P)\,\frac{e^{\frac{\mathrm{Pe}_e \frac{\delta x_e}{2}}{\delta x_e}} - 1}{e^{\mathrm{Pe}_e} - 1}\right) - \Gamma_e\left((\phi_E - \phi_P)\,\frac{\mathrm{Pe}_e}{\delta x_e}\,\frac{e^{\frac{\mathrm{Pe}_e x}{\delta x_e}}}{e^{\mathrm{Pe}_e} - 1}\right)$$

$$= F_e\left(\phi_P + (\phi_E - \phi_P)\,\frac{e^{\frac{\mathrm{Pe}_e}{2}} - 1}{e^{\mathrm{Pe}_e} - 1}\right) - F_e\,(\phi_E - \phi_P)\,\frac{e^{\frac{\mathrm{Pe}_e}{2}}}{e^{\mathrm{Pe}_e} - 1}$$

$$\boxed{J_e = F_e\left(\phi_P + \frac{\phi_P - \phi_E}{e^{\mathrm{Pe}_e} - 1}\right)}$$

Exponential Method

$$J_e = (\rho u)_e\,\phi_e - \Gamma_e\left(\frac{d\phi}{dx}\right)_e$$



$$J_e = F_e\left(\phi_P + (\phi_E - \phi_P)\,\frac{e^{\frac{\mathrm{Pe}_e \frac{\delta x_e}{2}}{\delta x_e}} - 1}{e^{\mathrm{Pe}_e} - 1}\right) - \Gamma_e\left((\phi_E - \phi_P)\,\frac{\mathrm{Pe}_e}{\delta x_e}\,\frac{e^{\frac{\mathrm{Pe}_e x}{\delta x_e}}}{e^{\mathrm{Pe}_e} - 1}\right)$$

$$= F_e\left(\phi_P + (\phi_E - \phi_P)\,\frac{e^{\frac{\mathrm{Pe}_e}{2}} - 1}{e^{\mathrm{Pe}_e} - 1}\right) - F_e\,(\phi_E - \phi_P)\,\frac{e^{\frac{\mathrm{Pe}_e}{2}}}{e^{\mathrm{Pe}_e} - 1}$$

$$\boxed{J_e = F_e\left(\phi_P + \frac{\phi_P - \phi_E}{e^{\mathrm{Pe}_e} - 1}\right)}$$ Similarly we have $$\boxed{J_w = F_w\left(\phi_W + \frac{\phi_W - \phi_P}{e^{\mathrm{Pe}_w} - 1}\right)}$$

Exponential Method

$$\frac{d}{dx}\left(\rho u\phi - \Gamma\frac{d\phi}{dx}\right) = 0 \longrightarrow J_e - J_w = 0$$

$$F_e\left(\phi_P + (\phi_P - \phi_E)\frac{1}{e^{\mathrm{Pe}_e} - 1}\right) - F_w\left(\phi_W + (\phi_W - \phi_P)\frac{1}{e^{\mathrm{Pe}_w} - 1}\right) = 0$$

Exponential Method

$$\frac{d}{dx}\left(\rho u\phi - \Gamma\frac{d\phi}{dx}\right) = 0 \longrightarrow J_e - J_w = 0$$

$$F_e\left(\phi_P + (\phi_P - \phi_E)\frac{1}{e^{\mathrm{Pe}_e} - 1}\right) - F_w\left(\phi_W + (\phi_W - \phi_P)\frac{1}{e^{\mathrm{Pe}_w} - 1}\right) = 0$$

$$a_P\phi_P = a_E\phi_E + a_W\phi_W$$

By comparing these two equations,
the analytical coefficients can be found as follows:

$$a_E = \frac{F_e}{e^{\mathrm{Pe}_e} - 1} \qquad a_W = \frac{F_w e^{\mathrm{Pe}}}{e^{\mathrm{Pe}_e} - 1} \qquad a_P = a_E + a_W$$

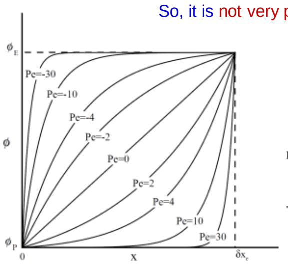
This method is called Exponential Method

Exponential Method

Despite high accuracy and physical behavior,

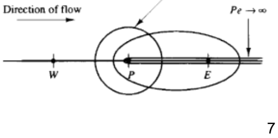
the exponential method has large computational cost.

So, it is not very popular.



Pe=0 Pure Diffusion

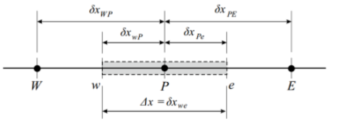
Pe=∞ Pure Convection



Example

Check the satisfaction of 4 rules for Central difference method.

$$\frac{d}{dx}\left(\rho u\phi - \Gamma\frac{d\phi}{dx}\right) = 0$$



Using central difference method, we have

$$(\rho u)_e\frac{\phi_E + \phi_P}{2} - (\rho u)_w\frac{\phi_P + \phi_W}{2} = \frac{\Gamma_e(\phi_E - \phi_P)}{\delta x_e} - \frac{\Gamma_w(\phi_P - \phi_W)}{\delta x_w}$$

Example

$$(\rho u)_e \frac{\phi_E + \phi_P}{2} - (\rho u)_w \frac{\phi_P + \phi_W}{2} = \frac{\Gamma_e(\phi_E - \phi_P)}{\delta x_e} - \frac{\Gamma_w(\phi_P - \phi_W)}{\delta x_w}$$

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

$$a_W = D_w + \frac{F_w}{2} \qquad a_E = D_e - \frac{F_e}{2} \qquad a_P = a_E + a_W + F_e - F_w$$

The above discretization is second order

Example

$$(\rho u)_e \frac{\phi_E + \phi_P}{2} - (\rho u)_w \frac{\phi_P + \phi_W}{2} = \frac{\Gamma_e(\phi_E - \phi_P)}{\delta x_e} - \frac{\Gamma_w(\phi_P - \phi_W)}{\delta x_w}$$

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

$$a_W = D_w + \frac{F_w}{2} \qquad a_E = D_e - \frac{F_e}{2} \qquad a_P = a_E + a_W + F_e - F_w$$

Suppose that $\phi_W = 100, \quad \phi_E = 200, \quad D = \frac{\Gamma}{\delta x} = 1, \quad F = \rho u = 4$

We expect that $100 < \phi_P < 200$

Example

$$(\rho u)_e \frac{\phi_E + \phi_P}{2} - (\rho u)_w \frac{\phi_P + \phi_W}{2} = \frac{\Gamma_e(\phi_E - \phi_P)}{\delta x_e} - \frac{\Gamma_w(\phi_P - \phi_W)}{\delta x_w}$$

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

$$a_W = D_w + \frac{F_w}{2} \qquad a_E = D_e - \frac{F_e}{2} \qquad a_P = a_E + a_W + F_e - F_w$$

Suppose that $\phi_W = 100, \quad \phi_E = 200, \quad D = \frac{\Gamma}{\delta x} = 1, \quad F = \rho u = 4$

$$a_E = 1 - 2 = -1, \quad a_W = 3, \quad a_P = 2$$

?

$$2\phi_P = -\phi_E + 3\phi_W = 300 - 200 = 100 \Longrightarrow \phi_P = 50!$$

Non-Physical Solution

Central Difference Method

Hints:

- Non-physical solution occurs due to negative coefficients which contradicts the second law.
- Central differenece method is suitable for low Peclet numbers ($Pe = \frac{F}{D}$)

In the example, Pe is: $\begin{cases} D = \frac{\Gamma}{\delta x} = 1 \\ F = \rho u = 4 \end{cases}$ $\quad \text{Pe} = \frac{F}{D} = \frac{4}{1} = 4$

Central Difference Method

To have physical solution, all neighbor coefficients should be positive.

Central Difference Method

$$a_W = D_w + \frac{F_w}{2}$$
$$a_E = D_e - \frac{F_e}{2}$$

$$\boxed{a_W = D_w + \frac{F_w}{2} \geq 0 \Rightarrow \frac{F_w}{D_w} = \mathrm{Pe}_w \geq -2}$$
$$\boxed{a_E = D_e - \frac{Fe}{2} \geq 0 \Rightarrow \frac{F_e}{D_e} = \mathrm{Pe}_e \leq 2}$$

$$\rightarrow |\mathrm{Pe}| \leq 2$$

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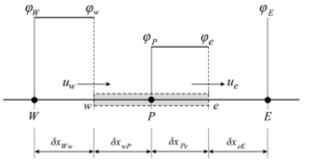
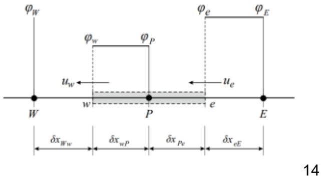
Upwind Method

Upwind method is used for high Peclet numbers (Pe)



The value on the cell face is equal to the upstream node

$$\phi_e = \begin{cases} \phi_P & F_e > 0 \\ \phi_E & F_e < 0 \end{cases}$$



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Upwind Method

Advantage: The upwind method does not produce oscillation results

Disadvantage: The upwind method is numerically diffusive

Using Taylor series in Cartesian coordinate system and Fe>0, we have

$$\phi_e = \phi_P + (x_e - x_P)\left(\frac{\partial \phi}{\partial x}\right)_P + \frac{(x_e - x_P)^2}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_P + O(h^3)$$

The upwind method uses only the first term

$$f_e^d = \Gamma_e \left(\frac{\partial \phi}{\partial x}\right)_e$$

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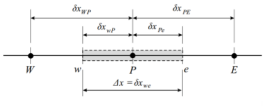
Upwind Method

Consider 1-D Convection-Diffusion equation

$$\frac{d}{dx}\left\{\rho u \phi - \Gamma \frac{d\phi}{dx}\right\} = 0$$



On the right face of cell element we have

$$\frac{d}{dx}\left\{(\rho u)_e \phi_e - \Gamma_e \left(\frac{d\phi}{dx}\right)_e\right\} = 0$$

$$\phi_e = \phi_P + (x_e - x_P)\left(\frac{\partial \phi}{\partial x}\right)_P + \frac{(x_e - x_P)^2}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_P + O(h^3) \qquad x_e - x_P = \Delta x/2$$

$$\left(\frac{d\phi}{dx}\right)_e = \left(\frac{d\phi}{dx}\right)_P + \frac{\Delta x}{2}\left(\frac{d^2\phi}{dx^2}\right)_P + \cdots$$

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Upwind Method

$$\frac{d}{dx}\left\{(\rho u)_e\phi_P - \left[\Gamma_e - \frac{\Delta x}{2}(\rho u)_e\right]\left(\frac{d\phi}{dx}\right)_e\right\} = 0$$

$$\Gamma_e^{num} = (\rho u)_e \Delta x/2$$ Numerical Diffusion

Hints:

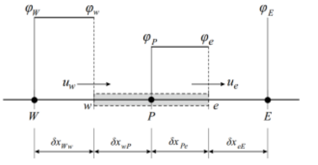
- In 2-D and 3-D problems in which flow direction is not perpendicular to the grid, the numerical diffusion increases
- Smaller grid can be used to increase the accuracy of results of upwind method

Upwind Method

$$\frac{d}{dx}\left\{\rho u\phi - \Gamma\frac{d\phi}{dx}\right\} = 0$$

$$\delta x_e = \delta x_w = \delta x$$

$u_w, u_e > 0$: $\phi_w = \phi_W$, $\phi_e = \phi_P$



$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$$

$$F_e\phi_P - F_w\phi_W = \frac{\Gamma_e}{\delta x}(\phi_E - \phi_P) - \frac{\Gamma_w}{\delta x}(\phi_P - \phi_W)$$

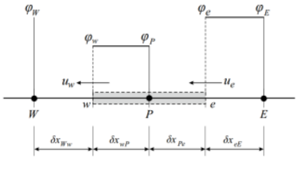
$$\Rightarrow [(F_w + D_w) + D_e + (F_e - F_w)]\,\phi_P = (F_w + D_w)\,\phi_W + D_e\phi_E$$

Upwind Method

$$\frac{d}{dx}\left\{\rho u\phi - \Gamma\frac{d\phi}{dx}\right\} = 0$$

$$\delta x_e = \delta x_w = \delta x$$

$u_w, u_e < 0$: $\phi_w = \phi_P$, $\phi_e = \phi_E$



$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$$

$$F_e\phi_E - F_w\phi_P = \frac{\Gamma_e}{\delta x}(\phi_E - \phi_P) - \frac{\Gamma_w}{\delta x}(\phi_P - \phi_W)$$

$$\Rightarrow [D_w + (D_e - F_e) + (F_e - F_w)]\,\phi_P = D_w\phi_W + (D_e - F_e)\,\phi_E$$

Upwind Method

In general, we can use the following relations for upwind method

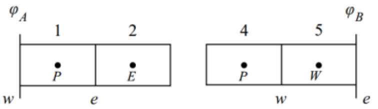
$$a_E = D_e + \max[-F_e, 0]$$

$$a_W = D_w + \max[F_w, 0]$$

$$a_P = a_E + a_W + (F_e - F_w)$$

Example

$$\frac{d}{dx}\left\{\rho u\phi - \Gamma\frac{d\phi}{dx}\right\} = 0$$



$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$$

$$u > 0 \qquad \phi_e = \phi_P \qquad \phi_w = \phi_W$$

$$F\phi_P - F\phi_W = D(\phi_E - \phi_P) - D(\phi_P - \phi_W)$$

$$\phi_P(F + 2D) = \phi_W(F + D) + D\phi_E$$

Example

Node 1

$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$$

$$F\phi_P - F\phi_A = D(\phi_E - \phi_P) - 2D(\phi_P - \phi_A)$$

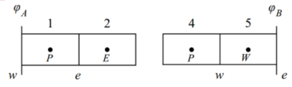
$$\Rightarrow (F + 3D)\phi_P = D\phi_E + (F + 2D)\phi_A$$

Node 5

$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$$

$$F\phi_P - F\phi_W = 2D(\phi_B - \phi_P) - D(\phi_P - \phi_W)$$

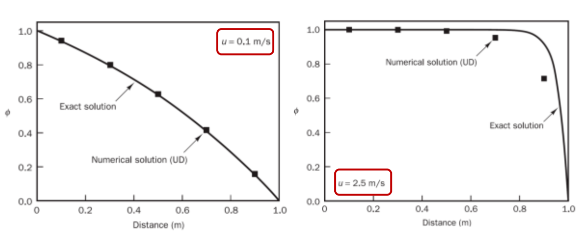
$$\Rightarrow (F + 3D)\phi_P = (F + D)\phi_W + 2D\phi_B$$



Example

S_u	S_P	a_E	a_W	Node
$(2D + F)\phi_A$	$-(2D + F)$	D	0	1
0	0	D	$D + F$	2-4
$2D\phi_B$	$-2D$	0	$D + F$	5

Example



Hybrid Method

Hybrid method is the combination of Central Difference and Upwind method

Hybrid Method
- $|\mathrm{Pe}| \le 2$ Central Difference
- $|\mathrm{Pe}| \ge 2$ Upwind

$$\mathrm{Pe}_w = \frac{F_w}{D_w} = \frac{(\rho u)_w}{\Gamma_w/\delta x_{WP}} \qquad q_w = (\rho u\phi - \Gamma\frac{d\phi}{dx})_w$$

$$q_w = F_w\left[\frac{1}{2}\left(1+\frac{2}{\mathrm{Pe}_w}\right)\phi_W + \frac{1}{2}\left(1-\frac{2}{\mathrm{Pe}_w}\right)\phi_P\right] \qquad -2 < \mathrm{Pe}_w < 2$$

$$q_w = F_w\phi_W \qquad \mathrm{Pe}_w \ge 2$$

$$q_w = F_w\phi_P \qquad \mathrm{Pe}_w \le -2$$

Hybrid Method

$$a_P\phi_P = a_W\phi_W + a_E\phi_E$$

$$a_P = a_W + a_E + (F_e - F_w)$$

$$a_W = \max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$$

$$a_E = \max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$$

Power Law Method

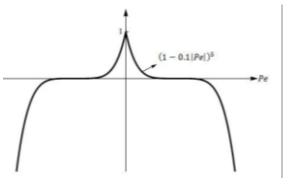
In the Power-Law method, a fifth order curve fit is used instead of exponential function in analytical solution. Therefore, high accurate results can be obtained and also, the computational cost decreases dramatically.

$$a_P\phi_P = a_E\phi_E + a_W\phi_W$$

$$a_E = D_e \ \max[0, (1-0.1|\mathrm{Pe}|)^5] + \max[0, -F_e]$$

$$a_W = D_w \ \max[0, (1-0.1|\mathrm{Pe}|)^5] + \max[0, F_w]$$

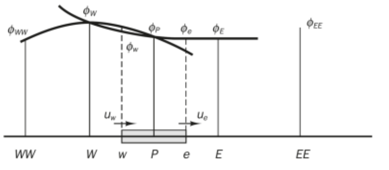
$$a_P = a_W + a_E + (F_e - F_w)$$



Quick Method

Quick method uses a three-point upstream-weighted quadratic interpolation for cell face values. The face value of φ is obtained from a quadratic function passing through two bracketing nodes (on each side of the face) and a node on the upstream side

$$\phi_e = \begin{cases} \phi_P + g_1(\phi_E - \phi_P) + g_2(\phi_P - \phi_W) & u > 0 \\ \phi_E + g_3(\phi_P - \phi_E) + g_4(\phi_E - \phi_{EE}) & u < 0 \end{cases}$$



Quick Method

$$\phi_e = \phi_P + (x_e - x_P)\left(\frac{d\phi}{dx}\right)_P + \frac{(x_e - x_P)^2}{2!}\left(\frac{d^2\phi}{dx^2}\right)_P$$

$$\phi_E = \phi_P + (x_E - x_P)\left(\frac{d\phi}{dx}\right)_P + \frac{(x_E - x_P)^2}{2!}\left(\frac{d^2\phi}{dx^2}\right)_P \qquad u > 0$$

$$\phi_W = \phi_P + (x_W - x_P)\left(\frac{d\phi}{dx}\right)_P + \frac{(x_W - x_P)^2}{2!}\left(\frac{d^2\phi}{dx^2}\right)_P$$

$$x_e - x_P = g_1 (x_E - x_P) - g_2 (x_W - x_P) \qquad g_1 = \frac{(x_e - x_P)(x_e - x_W)}{(x_E - x_P)(x_E - x_W)}$$

$$\frac{(x_e - x_P)^2}{2!} = g_1\frac{(x_E - x_P)^2}{2!} - g_2\frac{(x_W - x_P)^2}{2!} \qquad g_2 = \frac{(x_e - x_P)(x_E - x_e)}{(x_P - x_W)(x_E - x_W)}$$

Uniform Cartesian Grid $\quad \phi_e = \frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}\phi_W - \frac{9(\Delta x)^3}{16}\left(\frac{\partial^3\phi}{\partial x^3}\right)_P + O(h^4)$

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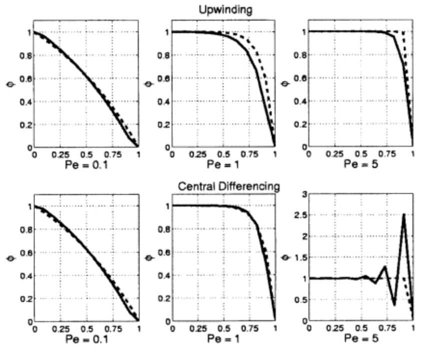
Example



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Higher Order Methods

$$\phi(x) = a_\circ + a_1 x + a_2 x^2 + a_3 x^3$$

$$\left(\frac{\partial\phi}{\partial x}\right)_e = a_1 + 2a_2 x + 3a_3 x^2$$

Uniform Cartesian grid

$$\phi_e = \frac{27\phi_P + 27\phi_E - 3\phi_W - 3\phi_{EE}}{48}$$

$$\left(\frac{\partial\phi}{\partial x}\right)_e = \frac{27\phi_E - 27\phi_P + \phi_W - \phi_{EE}}{24\Delta x}$$

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General Formulation

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -\frac{\partial p}{\partial x} \quad , \quad \phi = u \qquad \text{X-Momentum}$$

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -\frac{\partial p}{\partial y} \quad , \quad \phi = v \qquad \text{Y-Momentum}$$

$$J_x = \rho u\phi - \Gamma\frac{\partial\phi}{\partial x} \qquad J_y = \rho v\phi - \Gamma\frac{\partial\phi}{\partial y}$$

$$J_e^* = \frac{J_e\,\delta x_e}{\Gamma} = \frac{(\rho u)_e\,\phi\,\delta x_e}{\Gamma} - \frac{\partial\phi}{\partial(x/\delta x_e)}$$

$$J_e^* = \mathrm{Pe}_e\,\phi - \frac{d\phi}{d(x/\delta x_e)}$$

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General Formulation

$$J_e^* = \mathrm{Pe}_e[\alpha\phi_P + (1-\alpha)\phi_E] - \beta[\phi_P - \phi_E]$$

α and β are non-dimensional coefficients that depend on Pe.

$$J_e^* = B\phi_P - A\phi_E \qquad A = -[\beta + \mathrm{Pe}_e(1-\alpha)] \qquad B = \alpha\mathrm{Pe}_e - \beta$$

$$J_e^* = \mathrm{Pe}_e\phi_P = \mathrm{Pe}_e\phi_E \qquad \text{If } \phi_E = \phi_P$$

$$J_e^* = (A + \mathrm{Pe}_e)\phi_P - A\phi_E \qquad \mathrm{Pe}_e = \frac{(\rho u)_e \delta x_e}{\Gamma_e} \text{ Or } \mathrm{Pe}_e = \frac{(\rho u)_e[\Delta y]}{(\Gamma_e/\delta x_e)[\Delta y]}$$

General Formulation

$$\mathrm{Pe}_e = \frac{F_e}{D_e} = \frac{\text{Convection}}{\text{Diffusion}}$$

$$J_e^* = \frac{J_e \delta x_e}{\Gamma_e} = \frac{J_e}{D_e} \qquad J_e^* = \frac{J_e}{D_e} = \frac{F_e}{D_e}\phi_P + A(\phi_P - \phi_E)$$

$$J_e - F_e\phi_P = D_e A(\phi_P - \phi_E)$$

$$J_e - F_e\phi_E = AD_e(\phi_P - \phi_E) \qquad F_e < 0$$

$$J_e - F_e\phi_P = AD_e(\phi_P - \phi_E) \qquad F_e > 0$$

$$\boxed{J_e - F_e\phi_P = a_E(\phi_P - \phi_E) \qquad a_E = AD_e + \max[-F_e, 0]}$$

General Formulation

$$\boxed{a_E = AD_e \qquad J_e - F_e\phi_P = AD_e[\phi_P - \phi_E] \qquad F_e > 0}$$

$$\boxed{\begin{aligned} a_E = AD_e - F_e \qquad J_e - F_e\phi_P &= (AD_e - F_e)(\phi_P - \phi_E) \qquad F_e < 0 \\ &= AD_e\phi_P - F_e\phi_P - AD_e\phi_E + F_e\phi_E \end{aligned}}$$

$$\boxed{\begin{aligned} a_E &= D_e A(|\mathrm{Pe}|) + \max[-F_e, 0] & a_N &= D_n A(|\mathrm{Pe}|) + \max[-F_n, 0] \\ a_W &= D_w A(|\mathrm{Pe}|) + \max[F_w, 0] & a_S &= D_s A(|\mathrm{Pe}|) + \max[F_s, 0] \\ a_P &= (a_E + a_W) + (F_e - F_w) \end{aligned}}$$

General Formulation

Method	$A(\lvert\mathrm{Pe}\rvert)$
Central Difference	$1 - 0.5\lvert\mathrm{Pe}\rvert$
Upwind	1
Hybrid	$\max[0, 1 - 0.5\lvert\mathrm{Pe}\rvert]$
Power Law	$\max[0, (1 - 0.1\lvert\mathrm{Pe}\rvert)^5]$
Exponential	$\lvert\mathrm{Pe}\rvert/\{\exp(\lvert\mathrm{Pe}\rvert) - 1\}$